# Tutorial 2 (27 Jan)

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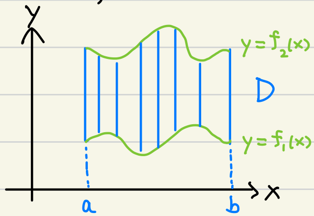
# Fubini's Theorem (Updated)

**Thm A** (Fubini's Theorem for continuous functions over Type I, II regions)

Let $f: D \to \mathbb{R}$ be a continuous function over a region $D$,

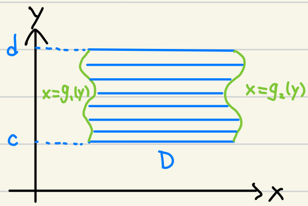



(I) If $D$ is of Type I,

i.e. $D = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b \,;\, f_1(x) \leq y \leq f_2(x)\}$,

where $f_1, f_2 : [a,b] \to \mathbb{R}$ are continuous with $f_1(x) \leq f_2(x)$, $\forall x \in [a,b]$

then $\iint_D f \, dA = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y) \, dy \, dx$.




(II) If $D$ is of Type II,

i.e. $D = \{(x,y) \in \mathbb{R}^2 \mid c \leq y \leq d \,;\, g_1(y) \leq x \leq g_2(y)\}$,

where $g_1, g_2 : [c,d] \to \mathbb{R}$ are continuous with $g_1(y) \leq g_2(y)$, $\forall y \in [c,d]$

then $\iint_D f \, dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y) \, dx \, dy$.

(III) If $D$ is of both Type I and Type II,

i.e. $D = \{a \leq x \leq b \,;\, f_1(x) \leq y \leq f_2(x)\} = \{c \leq y \leq d \,;\, g_1(y) \leq x \leq g_2(y)\}$

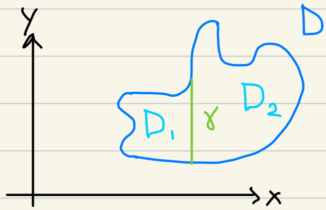
where $f_1, f_2 : [a,b] \to \mathbb{R}$, $g_1, g_2 : [c,d] \to \mathbb{R}$ are as in (I), (II),

then $\int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y) \, dy \, dx \overset{(I)}{=} \iint_D f \, dA \overset{(II)}{=} \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y) \, dx \, dy$.

# Additivity of multiple integrals over regions

**Thm B** Let $f: D \to \mathbb{R}$ be integrable over a region $D$, where $D = D_1 \cup D_2$ is a union of 2 subregions $D_1, D_2$ intersecting along a curve $\gamma$, then $f$ is integrable over $D_1, D_2$ with

$$\iint_D f\,dA \overset{\text{def}}{=} \iint_{D_1 \cup D_2} f\,dA \overset{\text{Thm}}{=} \iint_{D_1} f\,dA + \iint_{D_2} f\,dA.$$




**Rmk** More precise formulation of the above assumptions on $D$:

- $D = D_1 \cup D_2$, where $D_1, D_2 \subseteq D$ are subregions such that
- $D_1 \cap D_2 = \text{Image}(\gamma)$, where
- $\gamma: [0,1] \to D$ is piecewise $C^1$.

**Rmk** By applying the above theorem inductively, if

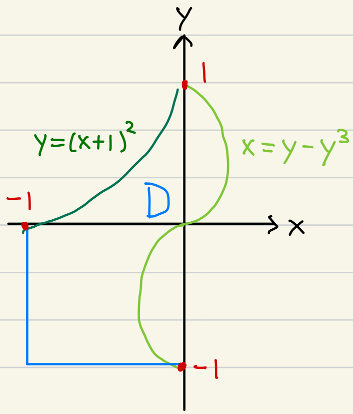
$D = D_1 \cup D_2 \cup \cdots \cup D_k$ is a finite union of subregions $D_i$

such that any 2 distinct subregions $D_i \neq D_j$ intersect along a curve (or empty)

then $f$ is integrable over $D_i$ for each $1 \leq i \leq k$ with

$$\iint_D f\,dA = \iint_{D_1} f\,dA + \iint_{D_2} f\,dA + \cdots + \iint_{D_k} f\,dA$$

**Ex** Evaluate $\iint_D y\,dA$, where $D$ is the region depicted as follows:



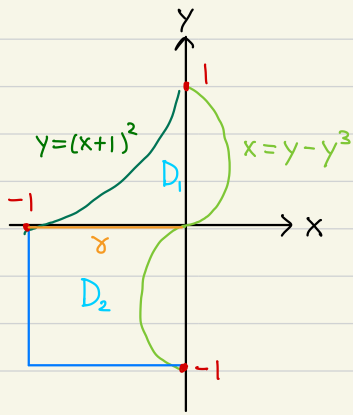

**Sol** Idea: Divide $D$ into Type I/II subregions

and evaluate $\iint_D y\,dA$ via Thm A, B.

**Step 1** Divide $D$ into Type I/II subregions.

For example: $D = D_1 \cup D_2$ as in $2^{nd}$ figure.

**Step 2** Describe $D_1, D_2$ as Type II subregions.




- $D_1 = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 1,\ \sqrt{y}-1 \le x \le y-y^3\}$

- $D_2 = \{(x,y) \in \mathbb{R}^2 \mid -1 \le y \le 0,\ -1 \le x \le y-y^3\}$

**Step 3** Apply Thm A (II) to evaluate $\iint_{D_i} y\,dA$.

- $\iint_{D_1} y\,dA = \int_0^1 \int_{\sqrt{y}-1}^{y-y^3} y\,dx\,dy = \int_0^1 y\left(y-y^3-(\sqrt{y}-1)\right)dy$

$$= \left[\frac{y^3}{3} - \frac{y^5}{5} - \frac{y^{\frac{5}{2}}}{\frac{5}{2}} + \frac{y^2}{2}\right]_0^1 = \frac{1}{3} - \frac{1}{5} - \frac{2}{5} + \frac{1}{2} = \frac{7}{30}.$$

- $\iint_{D_2} y\,dA = \int_{-1}^0 \int_{-1}^{y-y^3} y\,dx\,dy = \int_{-1}^0 y\left(y-y^3-(-1)\right)dy$

$$= \left[\frac{y^3}{3} - \frac{y^5}{5} + \frac{y^2}{2}\right]_{-1}^0 = 0 - \left(-\frac{1}{3} + \frac{1}{5} + \frac{1}{2}\right) = -\frac{11}{30}.$$

$\therefore$ By Thm B, $\iint_D y\,dA = \iint_{D_1} y\,dA + \iint_{D_2} y\,dA = \frac{7}{30} - \frac{11}{30} = -\frac{2}{15}$ //